A PLAN for IMPROVING the QUALITY of EXPOSITION in HIGH SCHOOL MATHEMATICS

"It is by means of names and numbers that the human understanding gains power over the world" Oswald Spengler in Decline of the West.

In order to raise the level of student achievement in secondary school mathematics, which everyone agrees is urgently necessary, there must be major improvements in the expository procedures employed by teachers. Accordingly, we specify the essential attributes of the teachers we need to bring about such improvements (Section I), offer some classroom-tested suggestions for their consideration (Section II), emphasize the current need to reinstate proof in secondary mathematics (Section III), outline the essential role of the mathematical community (Section IV) and describe certain external conditions that must exist in order for well-prepared teachers to be successful (Section V).

Section I: Essential Attributes of the Kind of High School Mathematics Teacher We Need.

A) Knows his subject. Has completed a strong undergraduate major in mathematics and, hopefully, has a minor in Physics or some other subject where mathematics is applied. Was especially attracted to mathematics by such courses as Algebraic Structures, Linear Algebra, Topology and the other "proof" courses" that follow calculus. Since many college majors no longer include a course in geometry such as that formerly derived from Coxeter (R 1) or Atschiller-Court (R 2), it is now necessary to assert that one of the most important of these "proof courses" should be a course in College Geometry. This course should

(1) Present advanced topics in Euclidean Geometry.

(2) Provide an introduction to Non-Euclidean Geometry.

(3) Stress "The Importance of Certain Concepts and Laws of Logic for the Study and Teaching of Geometry" Lazar (R 3).

{Reading this volume, which was Lazar's Ph.D. thesis, revolutionized my teaching. I hope that Nathan Lazar will, one day, be recognized as one of the great seminal thinkers in mathematics education.}

(4) Review and extend the student's understanding of the axiomatic method, in which proofs are forged by logic on a postulational base.

(5) Consider "incidence relations"and the danger of mak-ing unwarranted inferences from a drawing. (R 4)

(6) Provide experience with the highly instructive, open-ended construction problems found in the older geometry texts. (See Appendix 3)

The inclusion of such a required course would gradually remedy the existing intolerable situation where, very often, the teacher's knowledge of geometry does not extend beyond the covers of the high school text he is using!

A teacher who has not successfully completed these "proof courses" has no grasp of its ESSENTIAL and CHARACTERIZING properties of mathematics and IS NOT QUALIFIED TO TEACH MATHEMATICS IN HIGH SCHOOL.

B) Has taken courses in the Teaching of Mathematics which described various methods of presentation.

C) Has some knowledge about and great interest in the history of mathematics, including the contributions of the great mathematicians, the influence of the Greeks and the discovery of non-Euclidean geometry. (" A Concise History of Mathematics" by Dirk J Struik (R 5) provides an excellent introduction.)

Has perspectives gained by study of the history of mathematics education since 1900 which

*** Explores the various and sundry theories about how to teach school mathematics which have been promoted by our Schools of Education during this century.

*** Assesses the theories that Columbia University promulgated in the twenties and notes that some of these are now coming back under the banner of "Reform". [see "Orthodoxy Masquerading as Reform", E. D. Hirsch, Jr (R 6)].

*** Compares the texts used in the 30's and 40's with those used today.

*** Examines the reasons for the rise and fall of the "New Math" in the sixties and seventies (R 7)

*** Questions "Radical Constructivism" as the basis for NCTM sponsored reforms.

D) Loves mathematics because it is EXACT, ABSTRACT and LOGICALLY STRUCTURED. Considers these to be the ESSENTIAL and CHARACTERIZING properties of mathematics which enable it, when properly taught, to make unique and indispensable contributions to the education of all youth. Is determined to cherish these properties and believes that it is time to lead his high school students to understand and appreciate them.

Exactness is to be sought more than ever in this unforgiving digital world where even a dot out of place can destroy you. Moreover the student should realize that there are many situations in "real life" where there is literally no room for error in mathematical thinking.

The abstract quality of mathematics produces power by developing valid generalizations which are derived from many specific examples but are stated without reference to them and, hence, are applicable to many other specific cases.

Thus, the observation that $\frac{4+9}{2} > \sqrt{4 \cdot 9}$ and $\frac{7+13}{2} > \sqrt{7 \cdot 13}$ leads eventually to the general abstract statement $\frac{a+b}{2} > \sqrt{a \cdot b}$, *"The arithmetic mean of two different positive real numbers is greater than their geometric mean."*

The structured character of mathematics enables us to derive new facts (conclusions) from previously established facts (hypotheses) by building logical arguments (proofs). This proof process establishes connections between existing facts and, builds structure by adding to our fund of known facts. Proof, properly introduced, DOES NOT make mathematics more austere, forbidding and difficult. On the contrary, it can be an exciting game which provides the only path to understanding.

E) Is resolved to make mathematics interesting for his students not by making it easy (mathematics properly taught is difficult. (R 8)

not by making it continuous fun (The learning of mathematics requires a lot of hard work.)

not by down-playing its essential and characterizing properties because they are deemed to be too austere,

but by providing clear explanations which build the student's confidence by making mathematics seem reasonable. A mathematical concept, once understood is no longer intimidating. For example, the student who can explain why the $2ab$ term appears in the expansion of $(a+b)^2$ has acquired a satisfying understanding which liberates him from the frustrating process of trying to rely on memory to supply facts that he doesn't understand. For the student who must so rely, algebra is just a bag of soon-forgotten tricks or, as Henry James once said "A low form of cunning".

F) Is resolved to give due emphasis to the fact that the manifold APPLICATIONS of mathematics to the solution of practical problems make it a prerequisite to the successful study of virtually all branches of science. Will search for and spend some class time on "real world" problems whose solutions illuminate the mathematics in the course syllabus and which appeal to all or most of his students. Realizes that such problems hold great interest for students who continue to study mathematics for career reasons, most of whom follow geometry with a course in pre-calculus, It is also important for these students to realize that it is their understanding of the basic mathematical theory and techniques that will serve them best when it comes to applying mathematics to "real world" situations. Sometimes problems that have no apparent practical significance make surprising contributions to this understanding.

G) Teacher as director of learning. The high school mathematics teacher that we need considers the teacher's role to be that of "an instructor, one who imparts knowledge, a director of learning" (Webster) and intends to be a teacher in this traditional and, until recently, generally accepted sense. Chose teaching as a career because he believes with Professor Sylvia Feinburg that high school students "desperately need leadership guidance and stimulation from adults"(R 9). Expects to provide this leadership in the field of mathematics and to accept the responsibility and accountability that go with it. Regards exposition,"the art of presenting, explaining or expanding facts or ideas" as the mathematics teacher's principal function. (See Section II below). Sharply questions the romantic notion that high school students learn best when they are allowed to "discover" facts for themselves in cooperative learning (CL) sessions without direct instruction by the teacher. Feels that this demotes the teacher to the role of "Facilitator", a role he does not relish. ("Let me know when you need new batteries for your TI92.")

Is aware the CL is often defended on the grounds that it parallels practices used in business and industry, but sees a vast difference between a group of well-motivated professionals pooling their knowledge to solve a problem and a group of high school students working together to acquire basic knowledge that might be more efficiently acquired from direct instruction, Feels that occasaional use of CL, *in conditions where it is appropriate*, is enough to inculcate the idea of cooperation. In order to profit from direct instruction a student must learn to listen and to follow directions. These attributes are also essential for success in the business world. Would, nevertheless defend the more extensive use of CL by teachers who are convinced that it leads to better test results.

Understands that, as director of learning, he has the right to use any of the various methods of presentation at his command (see B above) Does not want specific methods of presentation such as CL prescribed for him. Such prescriptions constitute a misguided effort to standardize something which should not be standardized--the TEACHING of mathematics. Teaching is a highly individualistic art where each teacher must be free to use the methods of presentation that work best for him, as indicated by his student's performance on standardized tests.

H) Is resolved to be fair and objective in assigning course grades. Realizes that students are strongly motivated when they understand the grading system and believe that grades are objectively assigned. Believes that the course grade should be a valid measure of only one thing, the student's degree of mastery of the course syllabus. Considerations such as race, "equity", social status or the teacher's perception of the student's "mathematical disposition" should never affect the course grade.

H1) Plans to continue the traditional uses of the assessment system to diagnose learning difficulties and to provide information that can be used to improve instruction. (R 10)

H2) Eschews group testing because it (i) destroys the integrity of individual course grades (ii) impairs their validity as predictors of future performance and (iii) makes the assignment of individually prescribed remedial work impossible.

H3) Confident of his own ability to write objective tests that are valid measures of the student's understanding of the syllabus, readily accepts the premise that expert test writers can do the same. Would agree that the success of teaching and learning must be defined in terms of test results.

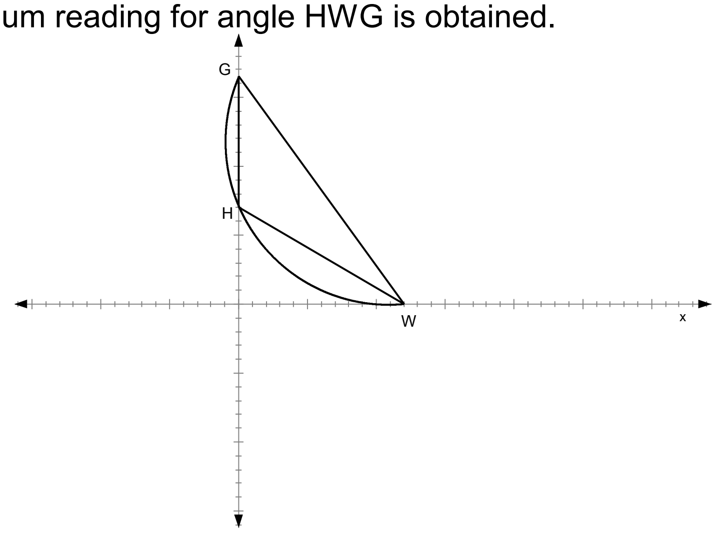
I) Recognizes the indispensable role of well-planned practice, repetition, review and drill in the development of the work and study habits that are necessary for the acquisition of mathematical knowledge. The student must not only understand routine algebraic operations, he must acquire facility in performing them--for himself without too early reliance on electronic devices. He must learn to perform routine tasks correctly and on time. This develops confidence, the feeling that "I can do that." It also frees the student's mind to deal with higher level challenges. After describing the "student skills" her sons acquired in Japanese schools, Carol Glick adds "Skills in the acquiring, may be uninspiring, but once possessed they are liberation itself."   (R 11)

While drill can be overdone, there is no essential conflict between concept building and intensive practice on mathematical operations. Each reinforces and illuminates the other.

J) Is highly proficient in the use of graphing calculators, computers and dynamic geometries such as Geometer's Sketchpad and Cabri Geometry II, which is now incorporated in the TI92. Intends to use these electronic devices to teach mathematics -- not vice versa. He regards the high school years as a time for developing understanding of the classical fundamentals (theorems) of mathematics. Students who have acquired this understanding will have no difficulty learning the "Training Manual" techniques necessary to operate successfully in the rapidly changing electronic world. Thousands have done so without formal instruction. Very few have learned mathematics that way. While he fully realizes that students are fascinated by the gleaming computer labs that are appearing in our more affluent communities, he harbors certain concerns.

J-1) Does not want students to regard the graphing calculator as an "authority" when dealing with proof. For example $\sin 2x = 2\sin x \cos x$, NOT because the graphing calculator gives the same graph for $y = \sin 2x$ and $y = 2\sin x \cos x$, but because we get this result when we let $y = x$ in our PREVIOUSLY PROVED formula $\sin(x+y) = \sin x \cos y + \sin y \cos x$. The student's reaction to the same graph phenomenon should be "Look, the TI92 got it right." The student must realize that it is not a proof.

J-2) Does not want dynamic geometries to be used to trivialize problems which are highly instructive when solved by classical methods. Example: We are asked to find the point W on the x- axis where GH subtends the maximum angle (see diagram). Using Sketchpad we can solve this problem by simply moving W along the x-axis until the maximum reading for angle HWG is obtained.



We don't have to understand how the theorem "*If from a point outside a circle, a tangent and secant are drawn, then the product of the secant and its external segment is equal to the square of the tangent,*" is involved. In fact, we don't have to understand anything. There are no connections with or appeals to any previous theorems. Since the student must make such connections and appeals in order to gain understanding of the structure of geometry, this Sketchpad solution is not satisfactory. This is a misuse of a dynamic geometry. Properly used, dynamic geometries enable the student to make exciting explorations and to generate conjectures whose truth value must be determined by applying the laws that govern mathematical proof.

K) Problem solving. Recognizes the pedagogical value of taking students "behind the scenes" by showing them how problems are CONSTRUCTED. This takes away some of the mystery and sheds some light on what problems are for. Moreover, problem construction is a direct operation whose difficult inverse is problem solving. Therefore, experience with problem construction should precede or at least accompany the student's encounter with problem solving.

L) Mathematics teacher as EXPOSITOR. The mathematics teachers we need are able to provide students with

Intensive training in the precise use of language and symbolism. This includes gradual mastery of 'The Language of Mathematical Exposition" (See Appendix 2)

A systematic introduction to PROOF which evolves from considering WHY certain statements are true or false.

The ability to state CONJECTURES whose truth value must be determined. This fosters the spirit of inquiry.

A sensitivity to inconsistency developed by testing conjectures and sometimes invoking counterexamples.

The ability to devise and investigate open-ended problems that evolve from the mathematics in the syllabus.

Section II: A Classroom Teacher's Suggestions for Consideration by the High School Mathematics Teachers We Need.

Before I present these suggestions let me say that I do not expect to be alone in this project. I expect a great deal of support from the many experienced teachers and from mature mathematicians who have recently taken the time to study the present state of instruction in school mathematics, as it has been affected by the NCTM Standards (See Section IV below). I believe that these mathematicians and teachers will strongly endorse the need for producing mathematics teachers with, substantially, the attributes set forth in Section I. I believe, too, that informed parents demand such teachers and intend to have them, even if they have to resort to private schools. If this is true, we have general agreement on our objectives. We can rest assured that the mathematical community will find ways to achieve them. It would be inappropriate for me, or anyone, to tell them how to do it. It is with this in mind that the following ideas and procedures are OFFERED FOR CONSIDERATION, with the hope that it will be found that they make some contribution to the art of mathematical exposition. They are derived from my forty-six years of experience as a teacher of mathematics in high school and college, which is briefly described in Appendix 1.

The reader must be prepared to encounter some concepts and procedures which are unusual in the sense that they do not, so far as I know, appear in any current texts.

Introduction to Proof

We begin with the implication $p \overset{?}{\Rightarrow} q$ which focuses attention on why one statement follows from another and thus furnishes the essential building block of proof. If p is true and we can supply a reason for the question mark, we have a proof of the statement q. *(Appendix 2, 17ii)*. The student should be trained to consider the converse and contrapositive forms of an implication. For example, the student should consider why the implication $(a = b) \Rightarrow (a^2 = b^2)$ and its contrapositive $(a^2 \neq b^2) \Rightarrow (a \neq b)$ are true while its converse $(a^2 = b^2) \Rightarrow (a = b)$ is false.

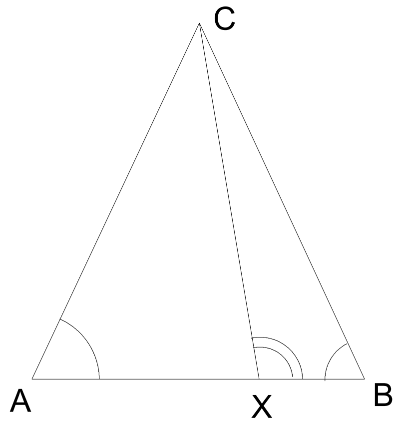
Now implications enable us to progress from the known to the unknown, and they can sometimes be linked together linearly thus: $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow e \Rightarrow f$. If each of these five implications is valid we have proof that $a \Rightarrow f$. Example: Solve $5 + \sqrt{x+7} = x$ for x :

$(5 + \sqrt{x+7} = x) \overset{(1)}{\Rightarrow} (\sqrt{x+7} = x - 5) \overset{(2)}{\Rightarrow}$
$(x + 7 = x^2 - 10x + 25) \overset{(3)}{\Rightarrow} (x^2 - 11x + 18 = 0) \overset{(4)}{\Rightarrow}$
$[(x-9)(x-2) = 0] \overset{(5)}{\Rightarrow} (x = 9 \vee x = 2)$ or that $\{x | 5 + \sqrt{x+7} = x\} \subseteq \{9, 2\}$.

We do not have a proof of the converse, $(x = 9 \vee x = 2) \Rightarrow (5 + \sqrt{x+7} = x)$ because the argument is not reversible. In addition to supplying the numbered reasons the student should explain why it is not reversible and determine that the solution set is $\{9\}$.

This example suggests that the study of proof should begin in algebra. Such proofs require reasons and, as we proceed, it gradually becomes evident that the reasons are supplied by the properties of an ordered field and theorems based on these properties. So, at some point in the algebra course the properties of a field should be listed as they were in *Pearson-Allen, A Logical Approach, 1964.*

Sometimes flow proofs involve bracketed statements. Example:
Prove: *The length of a line segment which joins the vertex of an isosceles triangle to a point in the interior of the base is less than the length of either leg*.



In terms of our figure we must prove:

$$\left.\begin{array}{c}\text{ABC is a }\triangle\\ \overline{\text{AC}}\cong\overline{\text{BC}}\\ \text{X is an internal point of }\overline{\text{AB}}\ (\text{X}\in\overline{\text{AB}})\end{array}\right\}\Rightarrow\ \text{CX} < \text{CB}$$

$$\left.\begin{array}{l}\left.\begin{array}{c}\text{ABC is a }\triangle\\ \text{X}\in\overline{\text{AB}}\end{array}\right\}\overset{(1)}{\Rightarrow}\ \angle\text{CXB is an exterior angle of }\triangle\text{CXA}\ \overset{(2)}{\Rightarrow}\ m\angle\text{CXB} > m\angle\text{CAB}\\ \left.\begin{array}{c}\text{ABC is a }\triangle\\ \overline{\text{AC}}\cong\overline{\text{BC}}\end{array}\right\}\overset{(3)}{\Rightarrow}\ \angle\text{CAB}\cong\angle\text{CBA}\ (\angle\text{CBX})\ \overset{(4)}{\Rightarrow}\ m\angle\text{CAB} = m\angle\text{CBX}\end{array}\right\}$$

$$\overset{(5)}{\Rightarrow} m\angle\text{CXB} > m\angle\text{CBX}\ \overset{(6)}{\Rightarrow}\ \text{CX} < \text{CB}$$

Why the flow format? Well, it is my position that beginning students need some way to understand the structure of a proof, and therefore should not begin with the nebulous "essay" form where it is hard for the teacher to provide a diagnostic analysis. This has long been recognized and one result was the "ledger" or two-column format. Indeed these have come to represent and characterize proof for many teachers. This is unfortunate because two-column proof in geometry has become a target of ridicule and this has served to discredit, in some measure, the whole idea of proof. Also it tends to exclude the study of proof from ninth grade algebra where it should begin and where it is much more easily presented. Ledger proofs are better than no proofs at all (which seems to be the alternative now), but they have serious defects. Clearly our building block $p \Rightarrow q$ does not fit into the two-column proof format. Moreover, two-column proofs do not show the structure or flow of the argument. When you get to step 10, just which of the previous steps are you counting on? I used to insert numbers and draw lines all over the proof in order to explain this. Most unsatisfactory. Then I saw a flow proof somewhere and began to use linkages of statement connected by implication symbols -- sometimes linear, sometimes bracketed -- so that the conjunction of several statements could imply a new statement. The effect was startling! Once I began using flow proofs, my students would not allow me to use anything else until they finally gained enough confidence to translate flow proofs to essay proofs, which, of course, was my objective in the first place.

To return to the example above, students should seldom be confronted with such a proof, unless it is for the purpose of supplying reasons in an open-book quiz situation. Instead, the proofs should be constructed in class with teacher and students cooperating in the construction. Sometimes we work backward from the conclusion in search of sufficient conditions. Sometimes we work forward from the hypothesis in search of necessary conclusions. We try to make the two lines of reasoning meet to form a seamless sequence from our hypothesis to our conclusion, i. e. a valid proof. If we succeed we have considerably enhanced our own understanding and we have a proof that other people can understand and one that is easily translated to essay form. In making this translation the student is not writing about mathematics, he is learning to use language precisely to write mathematics. One does not learn mathematics by writing about it anymore than one learns to play the piano by writing an essay about the *Moonlight Sonata.*

Note on Indirect Proof

Such arguments as *"Jim must have survived that plane crash because, if he had not, his name would be on the victim list, and it is not there,"* are commonly used in every day conversation. They have the form

$$\text{(iii)}\left.\begin{array}{c}(\sim p \Rightarrow q)\\ \sim q\end{array}\right\} \Rightarrow p$$

(*Appendix 2, 17iii*). We verbalize this as *"If a contradiction of a statement p implies a false statement, then p is true."* The clarifying point that (iii) is the second contrapositive of the direct argument

$$\text{(ii)}\left.\begin{array}{c}(\sim p \Rightarrow q)\\ \sim p\end{array}\right\} \Rightarrow q$$

is not included in most texts. It means that we can always replace an indirect proof with a direct proof of a contrapositive.

Use of contrapositives to clarify meaning and enhance the student's ability to communicate.

Consider the postulate *"Two points determine a line."* What does it mean? First we put it in bracket form letting A and B represent points and l_1 and l_2 represent lines.

$$\left.\begin{array}{ll} h_1: & A \in l_1\\ h_2: & B \in l_1\\ h_3: & A \in l_2\\ h_4: & B \in l_2\\ h_5: & A \neq B\end{array}\right\} \Rightarrow l_1 = l_2$$

Verbalized:

S_1 : *"If each of two lines contains the same two distinct points, the lines are equal."*

According to Lazar's definition of the multi-contrapositive (*Appendix 2, 18a*) the implication above has five contrapositives. As an exercise in the precise use of language we ask the student to verbalize each of these. It turns out that only two are distinct when verbalized. By exchanging $l_1 \neq l_2$ with $A \notin l_1$ we have

$$\left.\begin{array}{l} h_1:\ l_1 \neq l_2 \\ h_2:\ B \in l_1 \\ h_3:\ A \in l_2 \\ h_4:\ B \in l_2 \\ h_5:\ A \neq B \end{array}\right\} \Rightarrow A \notin l_1$$

Possible verbalization:

S_2 : *"If two distinct lines have a point in common, then any other point in one line is not in the other line."*

If we exchange $l_1 \neq l_2$ with $A = B$ we have

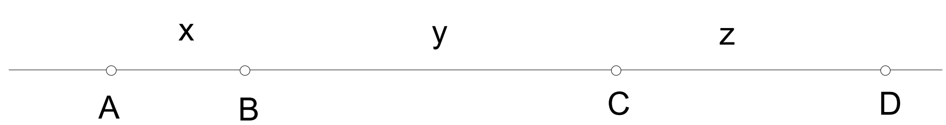
S_3 : *"If each of two points lie on both of two distinct lines then the points are equal."*

Now even though these statements are logically equivalent, they are significantly different: S_1 gives us a way to prove two lines equal, S_3 a way to prove two points equal, and S_2 a way to prove a point is not in a line. Taken together they completely exploit the meaning of the postulate.

We consider one more important example. The exterior angle theorem implies that: "If two lines are not parallel, and are cut by a transversal, then the alternate interior angles are not congruent." Contrapositive: "If the alternate interior angles are congruent, when two lines are cut by a transversal, then the lines are parallel." Since we know how to construct an angle congruent to another angle (Appendix 3 number 4 under Basic Constructions), it is easy to prove that through a point outside a line there exists a line parallel to it. But the student should be informed that it is impossible to prove Euclid's fifth postulate which asserts that there is only one such line. In fact centuries-long efforts to prove Euclid's fifth, using only the other postulates, led eventually to the discovery of non-Euclidean geometries, where Euclid's fifth does not hold. This is a spectacular triumph of logic over intuition.

Use of the converse to generate conjectures.

I first became interested in this idea when I encountered Euler's relation for four collinear points. Let A-B-C-D indicate that we have four collinear points in the order shown.



(|AB| $=$ x represents the distance between points A and B.)

Then, (A-B-C-D) $\Rightarrow$ (|AB| $\cdot$ |CD| $+$ |BC| $\cdot$ |AD| $=$ |AC| $\cdot$ |BD|).

(*In simpler notation:* $x \cdot z + y(x + y + z) = (x + y)(y + z)$).

Is the converse true? I thought it was true, but could not prove it. Then it occurred to me that the expression in the conclusion is satisfied when A, B, C, and D are the vertices of a square, or a rectangle, or an isosceles trapezoid, or indeed any cyclic quadrilateral (Ptolemy). At that point in my high school teaching career I began to use the multi-converse (*Appendix 2, 18ii*) to generate conjectures which must either be proved or disproved by citing a counter example.

Example 1: Given triangles ABC and $A'B'C'$ investigate the converses of

$$\text{(SAS)}\quad \left.\begin{array}{l} h_1:\ \overline{AC} \cong \overline{A'C'} \\ h_2:\ \angle C \cong \angle C' \\ h_3:\ \overline{CB} \cong \overline{C'B'} \end{array}\right\} \Rightarrow \left\{\begin{array}{rl} \angle A \cong \angle A' & : c_1 \\ \overline{AB} \cong \overline{A'B'} & : c_2 \\ \angle B \cong \angle B' & : c_1 \end{array}\right.$$

(Standard notation. *Appendix 3*)
Making this investigation the student discovers that all the triangle congruence theorems belong to a multi-converse set which also includes two non-theorems, SSA and AAA.

Example 2: In $\triangle ABC$ we have:

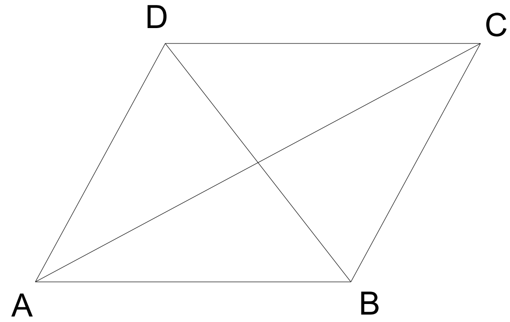
$$h:\ \overline{AC} \cong \overline{BC} \Rightarrow \left\{\begin{array}{rl} \angle A \cong \angle B & : c_1 \\ h_a \cong h_b & : c_2 \\ m_a \cong m_b & : c_3 \\ t_a \cong t_b & : c_4 \end{array}\right.$$

We ask the student to state each of the converses and determine its truth value. If we interchange h and c_2 the statement can be "*If two altitudes of a triangle are congruent, then the triangle is isosceles.*"

The fourth converse, whose hypothesis is $t_a \cong t_b$ is quite challenging. Teachers who use this procedure must expect to encounter many challenging problems not found in any text.

AT the end of the unit on parallelograms we use this example to lead the student on an explaration that will unify and extend his knowledge.

Example 3: In quadrilateral ABCD



$$\left.\begin{array}{l} h_1 : \overline{AB} \parallel \overline{DC} \\ h_2 : \overline{AD} \parallel \overline{BC} \end{array}\right\} \Rightarrow \left\{\begin{array}{rl} \overline{AB} \cong \overline{DC} & : c_1 \\ \overline{AD} \cong \overline{BC} & : c_2 \\ \angle A \cong \angle C & : c_3 \\ \angle B \cong \angle D & : c_4 \\ \overline{AC}\, bisects\, \overline{BD} & : c_5 \\ \overline{BD}\, bisects\, \overline{AC} & : c_6 \end{array}\right.$$

According to Lazar's definition (*Appendix 2, 18b*) there are $C_2^8 - 1 = 27$ converses. The student is asked to state each of these conjectures, and determine the truth value of those that are distinct when verbalized.

Consider the converse whose hypothesis is the conjunction of c_3 and c_6.

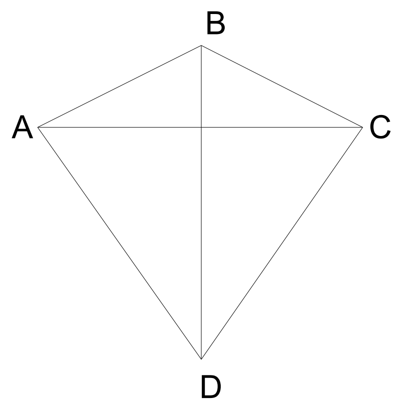
$$\left.\begin{array}{c} \angle A \cong \angle C \\ \overline{BD}\, bisects\, \overline{AC} \end{array}\right\} \overset{?}{\Rightarrow} ABCD \text{ is a parallelogram}$$

Possible statement: "*If two opposite angles of a quadrilateral are congruent and the diagonal joining their vertices is bisected by the other diagonal, then that quadrilateral is a parallelogram.*"

This conjecture is typical of those generated by Lazar's multi-converse definition in that:

1) its statement requires the careful use of language in a situation where the teacher knows what the student is trying to say, and
2) the decision regarding truth value is not immediately obvious.

In this case our hypothesis is satisfied by parallelogram $ABCD$, but a counter example is provided by a kite. So, the conjecture is false.



Problem formation (Problems from other Problems)

In order to describe this procedure I must again resort to specific examples. Our first example involves **allegation**: "The method of finding the value per pound of a mixture when the price per pound of each individual component is known." We deal with two components.

Example 1a: Find m, the value per pound of a mixture consisting of a pound of s-cent coffee (component 1) and b pounds of t-cent coffee (component 2). The solution develops the formula for the weighted mean, $m = \frac{as+bt}{a+b}$, which says that given a, b, s, and t, we can find m.

$$\left.\begin{array}{c} a \\ s \\ b \\ t \end{array}\right\} \Rightarrow m$$

Example 1b: Let $s = 40,\ b = 30,\ c = 60,\ and\, m = 52$, and restate Example 1a to find a (converse 1).

Possible statement: "*How many pounds of coffee worth 40 cents per pound must be mixed with 30 pounds of 60-cent coffee to produce a mixture worth 52 cents per pound?*"

Example 1c: Restate example 1a to find s. (Converse 2)

$$\left.\begin{array}{c} a \\ m \\ b \\ t \end{array}\right\} \Rightarrow s$$

Possible statement: "*A mixture contains 30 pounds of a brand worth 60 cents per pound and 20 pounds of another brand. If the mixture is worth 52 cents per pound, what is the per pound value of the other brand?*"

Example 1d: Prove: $(s < t) \Rightarrow (s < m < t)$

Example 1e: Generalize the formula in Example 1a for n components.

Question: What do we call a person who engages in allegations? Class?

Problems from formulas. The formula $A_n = P(1+r)^n$ represents the amount A_n which a principal of P dollars will accrue if invested at compound interest for n years at an annual interest rate of r.

Example 2: View the problem from the standpoint of a problem constructor.

Example 2a: Insert plausible values for three of the four variables and find the corresponding value for

a) A_n b) P c) r d) n

Example 2b: Prove the formula for A_n by mathematical induction.

The methods illustrated in this section have wide application at all levels of high school mathematics.

Section III. The need to reinstate proof as the culmination of exposition.

During roughly the first half of this century, this emphasis on proof-directed exposition would have been considered redundant by experienced teachers. In presenting proof they were seeking CLARITY IN EXPOSITION rather than pretentious rigor. Today, teachers trained in the last ten years in our Schools of Education may reject it as too rigorous, too austere, too "mathematical". Now *there is an urgent need to reinstate explanations that culminate in proof* at the secondary level. Let us review some of the evidence that supports this statement.

*** Proof has long been on the wane in secondary school mathematics. Older texts in geometry (circa 1920-60) placed great emphasis on proof by requiring students to prove many "originals" of all degrees of difficulty. Current texts seldom make such demands. This same trend toward downgrading proof to the point of elision is manifest in all sectors of the secondary curriculum. (Sadly, the NCTM has been too busy touting "Higher thinking skills" to notice.)

*** The employment of dynamic geometries such as Geometer's Sketchpad has been a boon, often replacing apathy with excitement, but, unless carefully directed by well-trained teachers, it does tend to blur the distinction between illustration and proof.

*** At a higher level the enormously complicated "proofs" generated by computers or, as in the case of the four-color problem, by vast linkages of computers, have served to make proofs less verifiable and to render the whole concept of proof more elusive. Some have seized on this as a reason for de-emphasizing axiomatic proof even at the research level. (R12), (R13)

*** NCTM's post-Standards policies no longer focus on improving exposition.

a) The NCTM has little to say about the need for improving the high school teacher's KNOWLEDGE OF MATHEMATICS without which improved exposition is impossible

b) The recent over-emphasis on cooperative learning, which seems to be based on a misinterpretation of "Constructivism" (R14) serves to deflect attention from what teachers do as expositors and focuses instead on whatever it is that "Facilitators" do.

*** The MAA's Task Force on the NCTM Standards expressed their views on this matter as follows: "One critical concern of the Task Force is the need for the Standards to more fully address issues of mathematical reasoning, the need for precision in mathematical discourse, and the role of proof." (R15)

*** In view of the miserable showing of US eighth grade math students on the TIMSS exams, (ranked 28th out of 41) the following quotes from the TIMSS videotape study, cited by Kim Mackey, are particularly significant. "It is likely that the kind of mathematics that students learn is related to the nature of the mathematics they are asked to study. Although constructing proofs and reasoning deductively are important aspects of mathematics, American students lacked opportunities to engage in these kinds of activities. None of the US lessons included proofs, whereas 10 percent of the German lessons and 53 percent of Japanese lessons included proofs." and "In a separate analysis of 30 lessons taken from each country by a group of experienced college teachers, 62 percent of Japanese lessons were found to contain deductive reasoning, compared to 21 percent in Germany and zero percent in the US."(R16)

Section IV. Essential Role of the Mathematical Community

Having established the need to improve exposition by reinstating proof we now turn to the essential role of the mathematical community in educating teachers who are able and eager to do this. While all members of CBMS should be involved, leadership should be provided by the college teachers of mathematics (MAA). Instruction designed to improve the exposition of high school mathematics must be closely geared to the subject matter of mathematics. This means that it cannot be left to our Schools of Education, but must involve the active participation of the mathematicians who teach at the college level. They must take some of the responsibility for improving the high school math instruction, whose failure has reached crisis proportions in some areas. (R17). They should help prepare the kind of teachers they want for their own children. Their own teaching should provide models of expository excellence, for the edification of the future teachers in their classes. (I was inspired by the presentations of Professor Allen T, Craig at the University of Iowa.) They alone have the deep understanding of mathematics which enables them to provide the lucid explanations that teacher trainees so urgently need. This requires effort, patience and a dedication to good teaching which some professors feel is often unrewarded. To open gates on the path to preferment one must publish research papers or obtain monetary grants. This often forces a concentration on non-teaching endeavors which may lead to neglect of exposition in the classroom.

In "A Mathematicians Apology", G.H. Hardy wrote "There is no scorn more profound, or on the whole, more justifiable, than that of the men who make for the men who explain. Exposition, criticism, appreciation is work for second-class minds". Considering the reward system now in place and their driving and commendable interest in research it is certainly understandable that many younger math department members may share Hardy's derogatory view of the expository process. They have reputations to build while still in their prime productive years, When it comes to improving exposition at the secondary level, we really can not expect much help from them. So we turn to their senior colleagues. These are mature mathematicians with well established reputations, who can now afford to focus their attention and indispensable expertise on the vitally important matter of improving exposition in secondary school mathematics by

1) Educating teachers who can accomplish this in the kind they want for their grandchildren (Section II) and

2) Establishing a situation in which these teachers can be successful

The first requires

(a) The reinstatement of a college course covering advanced topics in Euclidean geometry (see Section I above) and

(b) The development of a course in the teaching of secondary mathematics which considers the subject matter from an advanced standpoint, focuses on how to explain it and how to write tests that provide valid measures of the student's understanding.

The second requires

(a) Establishing national standards for measuring student competence in each high school course (algebra, geometry etc.) by providing standardized tests for each course, where individual scores can be compared with national and perhaps even world norms. These tests are to be externally graded, with individual grades widely publicized and made a part of the students permanent record. (See Bishop "The Power of External Standards". (R18)

(b) Providing a syllabus for each course which clearly delineates the mathematics that the student is expected to learn.

The establishment of national standards in school mathematics would be a long step toward maximizing the *common ground* held by our multiple and diverse cultures. This must be done if we want to live together in a harmonious and productive society. Also, it would help teachers deal with the "equity" problem, for it is the poor and disadvantaged who move frequently and, as a consequence, are victimized by the diverse standards generated by local control.

In dealing with these missions, committees of senior mathematicians, should seek input from high school math teachers who have a strong undergraduate major and some graduate work in mathematics and at least ten years of successful teaching experience. It would be highly desirable for the younger group of mathematicians to be represented on these committees, if this can be arranged.

Section V. External Conditions That Must Exist In Order for Well-Trained Teachers to be Successful These teachers must have

A) RESPECT. The well prepared high school mathematics teacher must be respected by parents, administrators and students, as one who is qualified to teach because he has successfully completed a RIGOROUS TRAINING PROGRAM, passed qualifying tests and keeps up to date by continuous study. His status and compensation should be on a par with that of an actuary or engineer. This will have the effect of making a career in the high school mathematics classroom attractive to our better students. Moreover, respected teachers, like respected coaches, are more effective instructors.

B) TIME. Mathematics teachers need TIME DURING THE SCHOOL DAY to plan presentations, confer with colleagues, write exams and diagnostic tests and grade student work as they have in other countries. In this regard the following statement by Linda Darling-Hammond is highly significant.

> "School systems in Germany, France, Switzerland, the Netherlands and Japan invest in a greater number of well-prepared, well paid and well supported teachers, rather than in a large bureaucracy populated by non-teaching staff hired to manage, inspect and control the work of teachers. Teacher education and ongoing professional development are much more extensive in these countries than in the United States; substantial time for learning and collegial work is built into the school day; and teachers make most school and curriculum decisions. This is fiscally possible because classroom teachers comprise 80 percent of the education employees in these countries, as compared to under 50 percent in the United States. By

investing in large administrative superstructures to control the work of teachers, rather than teach themselves, we have sucked resources out of classrooms where they could make a difference." (R 19)

Clearly, there must be a drastic reallocation of our financial resources in this country. If we want to give our well-trained math teachers a chance to succeed and are serious about meeting world standards, we must terminate this system where an employee's power, pay and prestige varies inversely with the number of classes taught and directly as the square of his distance from the classroom.

C) ADMINISTRATIVE SUPPORT. This involves

*** Providing teachers with swift and strong support in disciplinary matters. Even a Section I math teacher may occasionally need such support, and the knowledge that it is available will greatly enhance his effectiveness.

*** Insuring that each class assigned to a math teacher is a reasonably homogeneous group of students who have successfully completed all prerequisite courses. Yes, this means tracking and the end of "social promotion".

*** Creating a school atmosphere where academic achievement is respected; where it is "cool" to be smart; where class time is seldom interrupted and even the principal teaches a class or two in recognition of the fact that teaching is the most important thing and that the support of teaching is the administration's principal function.

*** Protecting math teachers from the pressures that generate grade inflation. This requires administrative endorsement of the external testing system described earlier. When students face a nationally administered standardized test, the teacher is no longer under pressure to inflate course grades. He becomes, instead, a coach or mentor who students need to help them meet an externally set and externally graded examination..

*** Maintaining the administrative arrangement where there is a separate MATHEMATICS DEPARTMENT, headed by a strong chairperson.

4) PARENTAL SUPPORT based on the understanding that learning mathematics requires hard work and nurtured by frequent parent-teacher contacts. *It is the parent's responsibility to send their children to school in a condition to learn. It is society's responsibility to insure that our schools are safe havens where students can concentrate on learning.* This must be

mentioned because improving exposition by providing well-trained (Section I) teachers will have little effect until these conditions are met.

In closing let me cite another statement in the "Mathematician's Apology" by G. H. Hardy. "My eyes were first opened by Professor Love who taught me for a few terms and gave me my first serious conception of analysis. But the great debt which I owe to him--was his advice to read Jordan's famous Cours d'analyze; and I shall never forget the astonishment with which I read this remarkable work--and learned for the first time--what mathematics really meant." Is this not a moving tribute to two great expositors? Does it not show that the"men who make" owe much to the "men who explain"? Good exposition can also "open eyes" and reveal what "mathematics really means", at the high school level. As the information age puts ever mounting pressure on our educational facilities, the work of the skilled expositor who can transmit knowledge and understanding to the next generation becomes increasingly important. There simply is not time to "reinvent the wheel" or to rediscover basic knowledge. If we want our math students to raise their level of performance and compete successfully with their peers in other industrialized countries, we must provide them with teachers who are skilled expositors and establish school situations where both teachers and students can succeed. Let's do it.

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(R10) Butler and Wren, "Teaching Secondary Mathematics" 1941. "To help provide more intelligent guidance of teaching and learning" and "To develop more effective curricula and educational experiences."

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(R12) JOHN HORGAN, "The Death of Proof", Scientific American, October 1993 pp.92-103.

(R13) Doron Zielburger, "Theorems for A Price-Tomorrow's Semi-Rigorous Mathematical Culture", Notices of the American Mathematical Society, October 1993 pp.978-981.

(R14) John R. Anderson, "Applications and Misapplications of Cognitive Psychology to Mathematics".

(R15) Ken Ross, Chair of the MAA Committee on the NCTM Standards in his report to the MAA Board of Governors. dated March 7, 1997, point 5 under "Reservations and Concerns about the Standards."

(R16) Mackey (mackeys@alaska.net)

(R17) Kenneth R, Weiss, writing in the Los Angeles Times for March 20, 1997 reports that " More than half the freshmen who entered the California University System last fall were unprepared for college level math". He also notes that "The percentage of unpreparsd students was the highest since Cal State began tracking the data in 1969."

(R18) John Bishop, "The Power of External Standards",Fall 1995 issue of The American Educator" .

(R19) Linda Darling-Hammond "Restructuring Schools for Student Success", Fall 1995 issue of DAEDALUS, p.158.